Near-Minimum Time Open-Loop Slewing of Flexible Vehicles

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Minimum time, open-loop, optimal controls are calculated for single-axis maneuvers of a flexible structure. By shaping the control profiles with two independent parameters, a wide variety of control histories can be produced. Based on the dynamics of the model, with a normalized time scale, the resulting Pontryagin's necessary conditions yield a nonlinear fixed final time, fixed final state, two-point boundary value problem with the maneuver time as a control parameter. Upon generating numerical solutions to the problem, the final maneuver time and residual flexural energy are compared to the bang-bang solution as a measure of the success of a given maneuver. Examples presented illustrate near-minimum time maneuvers with control of flexible modes in addition to the rigid body modes, as well as the qualitative and quantitative effect of the torque shaping parameters.

Introduction

NEAR-MINIMUM time attitude control of flexible spacecraft with active vibration suppression is a topic of current research. Although minimum time/fuel maneuvers have been previously examined,[1] such bang-bang controls cause spillover effects that may induce high residual flexural energy. The source of the energy spillover into the flexural modes is the instantaneous switching that results from the bang-bang controls, a potential problem for the control hardware (reaction wheels, control moment gyros, etc.) as well. Consequently, minimum-time optimal control problems are often of academic interest only for producing theoretical lower bounds for the maneuver time. We have also found that near bang-bang controls are usually very sensitive to model errors; therefore, control shaping is an important issue in obtaining robust controls.

Upon modifying the control profile with a smoothing function and transforming the independent variable (time), a near-minimum-time problem is generated with the mathematical form of a fixed-time nonlinear optimal control problem. The resulting boundary-value problem yields to a number of established methods of numerical solution. The controls are attenuated in such a way that the magnitude of the control rises smoothly from zero to the bounded maximum at the initiation of the maneuver and typically has an identically shaped reduction to zero at the final time. In addition, any instantaneous switch during the maneuver is shaped using a smooth, continuous function. The sharpness of the control trajectory is determined by a set of arbitrary parameters such that we can produce profiles with low control rates as well as controls that approach, to any desired degree, the bang-bang minimum time solution. The independent variable is transformed such that a linear free-final-time optimal control problem is converted to a nonlinear fixed-time problem, where the maneuver time is contained explicitly as a parameter in the transformed system. This augmented fixed-time problem can then be solved numerically for the controls and the corresponding minimum time required to complete the maneuver.

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The behavior of the system in response to the optimal controls specified by the set of parameters can be examined to determine tradeoffs necessary to accomplish certain mission objectives. For example, near bang-bang controls offer the shortest maneuver time; however, control spillover into the flexible modes will induce structural vibration, perhaps to an unacceptable degree. Conversely, a smoothly varying control reduces the spillover but may drastically increase the time required to complete the maneuver. We examine issues such as the minimum time required and residual structural energy for various representative control profiles.

Numerical examples for a low-order model of a flexible body are included to demonstrate the implementation of the algorithm and support our development of a general, effective, and practical approach to producing near-minimum time, open-loop control of flexible vehicles.

Minimum Time Optimal Control Problem

Consider a flexible body described by a set of linear, undamped, ordinary differential equations of motion of the form

$$M\ddot{x} + Kx = Gu \tag{1}$$

where M is an n by n positive definite mass matrix, x an n by 1 vector of configuration coordinates, K an n by n positive semidefinite stiffness matrix, G an n by m control influence matrix, and u an m by 1 control vector. The admissible controls $u(t)$ must satisfy saturation bounds $|u(t)| \leq u_{\max}$ [every element of $u(t)$ is bounded]. We shall transform Eq. (1) into a set of uncoupled equations in order to reduce the complexity of the optimal control problem. For low-order models, or models of discrete systems, this procedure may be unnecessary, although it will greatly simplify the coding effort. However, we must recognize that a discrete representation of any continuous system given by Eq. (1) involves some degree of approximation, which typically means that a number of the higher frequency modes will be inaccurately predicted by this model of the structure. Therefore, there is a practical reason for restricting our attention to controlling the modes (typically the lower half) that are accurately represented for the system under consideration.

By solving the algebraic eigenvalue problem associated with the unforced equations of motion for the diagonal eigenvalue matrix Λ and the eigenvector (modal) matrix E, the equations of motion may be uncoupled such that

$$\ddot{\eta} + \Lambda\eta = Bu \tag{2}$$

where

$$E^TME=I, \qquad E^TKE=\Lambda$$

$$x=E\eta, \qquad B=E^TG$$

$$|u_i(t)| \le u_{\max_i} \qquad i=1,2,\ldots,m$$

In the development of the near-minimum time optimal control problem, we shall seek to control only a subset of the modal coordinates, chosen from the lower frequency modes contained in Eq. (2). Since our work is primarily addressed to an examination of the effect of controlling the rigid body mode and the dominant flexible modes (as opposed to actively controlling all of the flexible body modes), the number of controlled modes in the examples shall be limited to three or less for illustrative purposes only. However, the derivation of the optimal control formulation contains no restrictions, and the number of controlled modes is discretionary.

To continue the derivation, we shall choose to control a set of N modes and form the vector z to represent the states of the controlled modal system, such that

$$\dot{z}=Az+Du, \qquad |u| \le u_{\max_i} \tag{3}$$

where

$$z=\mathrm{col}(\eta_c,\dot{\eta}_c) \qquad (2N \text{ by } 1)$$

$$A=\begin{bmatrix} 0 & I \\ -\Lambda_c & 0 \end{bmatrix} \qquad (2N \text{ by } 2N)$$

$$D=\begin{bmatrix} 0 \\ B_c \end{bmatrix} \qquad (2N \text{ by } m)$$

and η_c and $\dot{\eta}_c$ are the modal positions and velocities of the subset of modes that we choose to actively control. Spillover into the uncontrolled modes is a separate issue that we have addressed to date via numerical simulation studies. We seek the control that will drive the system from a given initial state to a specified target state

$$z(0)=\beta_0 \qquad Vz(t_f)=\beta_f \qquad (2N \text{ by } 1)$$

within the final time (t_f), at present unknown. The problem defined by Eq. (3) is a fixed final state, free final time, two-point boundary-value problem (TPBVP).[5] We shall transform this system into a fixed final state, fixed final time problem, which can be solved by a number of numerical methods,[2-4] by defining the normalized time variable

$$\tau=t/t_f \tag{4}$$

so that when $t=0$, $\tau=0$, and when $t=t_f$, $\tau=1$. Differentiating both sides of Eq. (4) leads to the identity

$$\frac{\mathrm{d}(\;)}{\mathrm{d}\tau}=t_f\frac{\mathrm{d}(\;)}{\mathrm{d}t} \quad \text{or} \quad (\mathring{\;})=t_f(\dot{\;}) \tag{5}$$

Using the identity and notation given in Eq. (5) and operating on Eq. (3), the fixed final time two-point boundary value problem is given by

$$\mathring{z}=t_f(Az+Du) \tag{6}$$

with the corresponding boundary conditions given by

$$z(\tau=0)=\beta_0 \qquad Vz(\tau=1)=\beta_f \tag{7}$$

The optimal control problem is to determine the controls applied to Eq. (6) that will satisfy these boundary conditions and minimize the performance index

$$J=\int_{t_0}^{t_f}\mathrm{d}t=\int_0^1 t_f\mathrm{d}\tau=t_f \tag{8}$$

The Hamiltonian, formed from the integrand of Eq. (8) and the right-hand side of Eq. (6), is

$$H=t_f+\lambda^T[t_f(Az+Du)] \tag{9}$$

where the $2N$ costates λ are a set of undetermined Lagrange multipliers. Pontryagin's principle, applied to Eq. (9), provides the equations governing the states and costates

$$\mathring{z}=\frac{\partial H}{\partial \lambda}=t_f(Az+Du) \tag{10a}$$

$$\mathring{\lambda}=-\frac{\partial H}{\partial z}=-t_fA^T\lambda \tag{10b}$$

and u is chosen such that Eq. (9) is minimized for all time. To accomplish this, the last term of Eq. (9) must be negative, a requirement that is mathematically stated for each element (u_i) of the control vector as

$$u_i=-u_{\max_i}\mathrm{sign}\left(\sum_{j=1}^{2N}D_{ji}\lambda_j\right) \tag{11}$$

where the subscripts indicate the elements of the corresponding vectors and matrices.

Equations (10) and (11) together with $H=0$ are the necessary conditions for the true minimum time optimal control problem. With certain exceptions (characterized by vanishing of the sign change in the controls), we note from Eq. (11) that the controls for a typical maneuver are instantaneously "switched" on and off (at $\tau=0$ and $\tau=1$) as well as for positive to negative at interior $(0<\tau<1)$ points.

Since the controls depend only on the sign of the argument in Eq. (11), the magnitude of the initial costates will not affect the saturation value $(\pm u_{\max})$ or the switch time determined for the controls. From Eq. (10), we see that the costates appear linearly in an unforced first-order, ordinary differential equation. Consequently, the costates do not have a unique magnitude, and we are free to impose any arbitrary condition that "normalizes" $\lambda(\tau)$ in some convenient way (see Ref. 5 for discussion of this point). We choose to introduce a constraint that the undetermined initial costates satisfy the normalization condition

$$\lambda^T(0)\lambda(0)=C \tag{12}$$

where C is an arbitrary positive constant. With this constraint, the search for suitable initial costates is restricted to a scaled hypersphere of initial conditions. The magnitude of C should be chosen such that the numerical behavior of the solution is enhanced which, for most applications, implies that $C \le 1$.

Profile Shaping of the Optimal Control

We propose to alter the switch functions given by Eq. (11) in such a way that the control applied to the system is smooth and continuous throughout the entire maneuver, with values between zero and $\pm u_{\max}$ (zero near the endpoints of the maneuver). An arbitrary change in the control profile violates the boundary condition $H(t_f)=0$ that should formally be satisfied as a necessary condition for optimal controls that produce a minimum time maneuver. In Ref. 5, we show that the arbitrary scaling on the costates could be adjusted to enforce $H(t_f)=0$, but this boundary condition is not needed and does not affect our solution. It is important to note that we seek suboptimal controls (near-minimum time) which eliminate the jump-discontinuities of the torque history in order to reduce structural vibrations. By turning to the suboptimal approach, we can "tune" the control profile in such a way that flexural energy considerations can be included within the framework of near-minimum time control formulations; that is, we can systematically trade off residual vibration with maneuver time. The arctangent can be used to approximate the switch functions and thereby develop control profiles that meet these criteria and approach, as a limiting case, the bang-bang controls determined by the sign function.

Consider the approximation of the sign function of a scalar variable to be expressed by

$$\operatorname{sign}[s(\tau)] \cong w(\tau,\Delta\tau)\frac{2}{\pi}\tan^{-1}\left[\frac{s(\tau)}{1-\alpha}\right] \tag{13}$$

where the arctangent function simulates the sign function[6] with the parameter α determining the degree of approximation, and, consequently, the sharpness of the switch and the multiplier $2/\pi$ is used to bound the arctangent between ± 1. As α approaches 1, the arctangent function in Eq. (13) approaches the ideal sign function; however, the approximation is smooth and continuous (as shown in Fig. 1 for several values of α). The multiplicative weight function $w(\tau,\Delta\tau)$, is introduced to "shape" the terminal behavior of the sign function approximation as necessary for specific applications. For example, we note that the arctangent approximation still has an "instant on-off" character, initially and finally, despite the fact that any interior switches are smooth and continuous. We can arbitrarily shape the terminal on-off conditions of Eq. (13) by choosing the weight function to attenuate the approximation at the endpoints ($\tau = 0,1$).

Let the weight function be given by

$$w(\tau,\Delta\tau) = \frac{1}{4}\left[\tanh\left(\frac{2r[\tau-(\Delta\tau/2)]}{\Delta\tau}\right)+1\right] \times\left[1-\tanh\left(\frac{2r[\tau-1+(\Delta\tau/2)]}{\Delta\tau}\right)\right] \tag{14}$$

where $\Delta\tau$ is a "rise time" for the function and r is a positive constant chosen such that the weight function obtains a prescribed value $[0 < w(\tau,\Delta\tau) < 1]$ for $\tau = \Delta\tau$ and $\tau = 1-\Delta\tau$. For our work we have used $r = 2.6$, which insures that $w(\Delta\tau,\Delta\tau) = w(1-\Delta\tau,\Delta\tau) = 0.9945$. This guarantees that the weight function is within 1% of maximum during the interior portion of the maneuver ($\Delta\tau \leq \tau \leq 1-\Delta\tau$). Although Eq. (14) appears complicated, it allows us to apply a single weight function to the approximation given by Eq. (13) with the effect of prescribing zero slope and zero magnitude to the endpoints of the sign approximation, as shown in Fig. 2. In addition, the rise time is a parameter that provides another method of tuning the control shape for specific mission requirements.

The approximation given by Eqs. (13) and (14) is smooth and continuous, with two parameters, α and $\Delta\tau$, available for shaping the terminal and interior switch profiles as required. Since this approximation is applied to a scalar variable, to introduce the approximation into the optimal control formulation requires that each element of the control vector be treated independently. Substituting the approximation from Eq. (13) into Eq. (11) gives the ith element of the control vector

$$u_i = -w(\tau,\Delta\tau)u_{\max_i}\frac{2}{\pi}\tan^{-1}\left(\sum_{j=1}^{2N} D_{ji}\lambda_i\right)\Big/(1-\alpha) \tag{15}$$

where $w(\tau,\Delta\tau)$ is given by Eq. (14). The near-minimum time optimal control problem is then given by Eqs. (10), (12), and (15), which includes the as yet undetermined final time t_f and the boundary conditions given by Eq. (7).

Near-Minimum Time Optimal Control Formulation

At this point, we have determined all of the equations and boundary conditions required to produce a minimum time, open-loop maneuver with control shaping. The optimal control problem is, as usual, a two-point boundary value problem with a partial set of initial conditions specified and a corresponding set of given final conditions. We form the near-minimum time optimal control problem by defining the augmented state vector

$$X = \operatorname{col}(z, t_f, \lambda) \qquad (4N+1 \text{ by } 1) \tag{16}$$

where the unknown final time is now a state variable. Obviously, Eq. (16) is also implicitly a function of α and $\Delta\tau$, but we suppress this dependence for notational compactness. We consider α and $\Delta\tau$ to be held fixed for each solution of Eq. (16); thus we have a two-parameter family of optimal solutions. The TPBVP may be written as the first order augmented state/costate equations

$$\mathring{X} = \begin{Bmatrix} \mathring{z} \\ \mathring{t}_f \\ \mathring{\lambda} \end{Bmatrix} = \begin{Bmatrix} t_f\left\{Az - w(\tau,\Delta\tau)\frac{2}{\pi}u_{\max}D\left[\tan^{-1}\left[\sum_{j=1}^{2N} D_{ji}\lambda_j\Big/\left(1-\alpha\right)\right]\right]\right\} \\ 0 \\ -t_f A^T\lambda \end{Bmatrix} \tag{17}$$

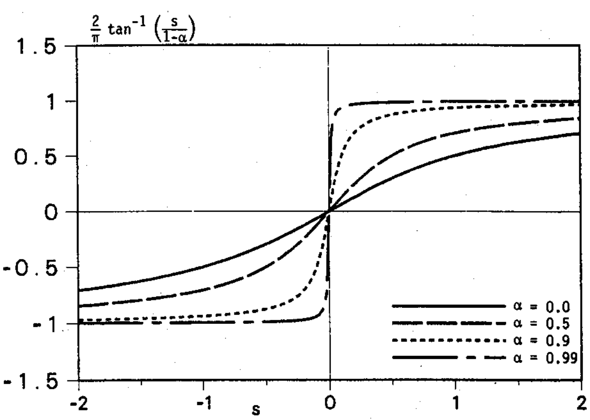

Fig. 1 Approximation of sign function.

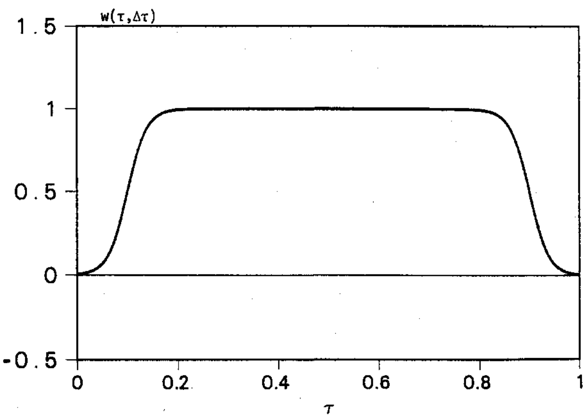

Fig. 2 Weight sanction applied to controls.

where by definition $t_f = X_{2N+1}$. The boundary conditions and constraints remain unchanged

$$z(0) = \beta_0 \qquad Vz(1) = \beta_f \tag{18a}$$

$$\lambda^T(0)\lambda(0) = C \tag{18b}$$

Note that Eq. (17) is a nonlinear function of the state variables, a consequence of the transformation given by Eq. (4). The optimal control problem has now been "reduced" to a nonlinear TPBVP of order $4N+1$ with $2N$ initial conditions, $2N$ final conditions, and one constraint specified. We treat the unknown maneuver time as a (0) state variable in the augmented system since our task is to solve Eq. (17) subject to Eqs. (18) for the unknown initial costates.

Because the problem is nonlinear, we must generally rely on numerical methods to complete the solution. We elect to solve the near-minimum time optimal control formulation through the use of the Method of Particular Solutions,[3] an iterative procedure for nonlinear equations, since the method relies on the principle of superposition. We must first linearize the equations governing the optimal control problem by using quasilinearization.[2] Let us define the right-hand side of Eq. (17) as a vector function f such that

$$\overset{\circ}{X} = f(X,\tau) \tag{19}$$

Expanding Eqs. (18b) and (19) in a Taylor's series to first order for the kth trial trajectory X_k gives

$$\overset{\circ}{X}_k + \Delta\overset{\circ}{X}_k = f(X_k,\tau) + \frac{\partial f(X_k,\tau)}{\partial X}\Delta X_k \tag{20a}$$

$$\lambda_k^T(0)\lambda_k(0) + 2\lambda_k^T(0)\Delta\lambda_k(0) = C \tag{20b}$$

in terms of the corrections $\Delta X_k(\tau)$ and $\Delta\lambda_k(0)$. Note that $\Delta\lambda_k$ is by definition a part of the departure motion vector ΔX_k. The corrected $k+1$th solution is then given by

$$X_{k+1} = X_k + \Delta X_k \tag{21}$$

Solving Eq. (21) for ΔX_k and substituting into Eq. (20) gives the quasilinearized optimal control formulation in terms of the $k+1$th trial solution

$$\overset{\circ}{X}_{k+1} = F_k(\tau)X_{k+1} + D_k(\tau) \tag{22}$$

$$z_{k+1}(0) = \beta_0, \qquad Vz_{k+1}(1) = \beta_f \tag{23a}$$

$$\lambda_k^T(0)\lambda_{k+1}(0) = \tfrac{1}{2}[C + \lambda_k^T(0)\lambda_k(0)] \tag{23b}$$

where we have defined

$$F_k(\tau) = \left.\frac{\partial f(X,\tau)}{\partial X}\right|_{X_k(\tau)} \quad (4N+1 \text{ by } 4N+1) \tag{24a}$$

$$D_k(\tau) = f(X_k,\tau) - F_k(\tau)X_k(\tau) \quad (4N+1 \text{ by } 1) \tag{24b}$$

The kth trial solution is known (we are solving for the $k+1$th solution) and, therefore, F_k, D_k, and λ_k are implicit functions of time τ.

The method of superposition assumes that the solution of the linearized equations governing the TPBVP can be written as a linear combination of a set of neighboring trajectories (found by choosing a judicious set of initial conditions for the unknown costates). Applying the boundary conditions to the expansion leads to a linear algebraic equation that can be solved for the correct initial costates of the $k+1$th solution. The procedure is then repeated until convergence is achieved.

Table 1 Spacecraft structural parameters

Hub mass	16 kg
Hub radius	0.3 m
Appendage stiffness	6.9×10^{10} N/m^2
Appendage density	2700 kg/m^3
Appendage length	2 m
Appendage height	0.1524 m
Appendage thickness	3.175×10^{-3} m
Tip mass	3 kg
Inertia of tip mass	8.51263×10^{-3} kg·m^2

Table 2 Natural frequencies

Mode no.	Natural frequency, rad/s
1	0.0
2	6.92
3	22.96
4	61.57
5	123.86

Application to a Low-Order System

To demonstrate the near-minimum time optimal control formulation, we apply the method to a simple structure that is composed of a rigid hub and one cantilevered flexible appendage where the appendage has a small mass attached to the tip. The hub is pinned to allow only a single-axis rotational degree of freedom, and the motion of the appendage is restricted to the plane perpendicular to the rotational axis. A single external torque is applied at the center of the hub. The spacecraft structural parameters are given in Table 1.

Using the Method of Assumed Modes,[7] where the mode shape is described by the function

$$\phi_i(x) = 1 - \cos\frac{i\pi x}{L} - \frac{1}{2}(-1)^i\left(\frac{i\pi x}{L}\right)^2 \tag{25}$$

the mass and stiffness matrices can be calculated. We shall restrict this model to a low-order ($n=5$) as defined in Eq. (1), so that the presentation of numerical results is manageable. Solving the algebraic eigenvalue problem produces the natural frequencies listed in Table 2.

For each of the numerical examples presented, the initial conditions are $\eta_1(0) = 1.22474$ (rigid body displacement of 5 deg), $\eta_{2\text{-}5}(0) = 0$, and $u_{max} = 5$.

Control of Mode 1

We shall first investigate near-minimum time maneuvers of the example problem with control of the rigid body mode being the only consideration. The true minimum time solution of the optimal control of a rigid body is well documented[5] and establishes the lower bound for all maneuvers. The time required to complete the bang-bang maneuver of the spacecraft considered in the following examples is determined to be 2.141 s. By adjusting the parameters (α and $\Delta\tau$) that govern the torque profile, we can widely vary the configuration of the optimal control and consequently cause large variations in the maneuver time and flexural energy. Note that since the residual energy varies during the application of the controls, the residual energy at the completion of the maneuver depends on the peak magnitude of the residual energy and the particular value of the energy at the completion of the maneuver.

In order to appreciate the degree to which the control can be manipulated, we shall examine rigid-body maneuvers using control profiles obtained through relatively large variations of α and $\Delta\tau$. In each case the maneuver time and residual structural energy are considered measures of the success of each single-axis maneuver, and each case will be quantitatively compared to the bang-bang maneuver that we found required 2.141 s to complete with an energy spillover (as measured by the final residual energy of the flexible modes) of $E=0.237$ Nm.

Case 1: $\alpha=99$, $\Delta\tau=0.1$

For the first case, we attempt to approximate the bang-bang slew with a sharp rise time (10% of the total maneuver time) and an extremely sharp central switch. The control profile generated in the near-minimum time problem for this example is shown in Fig. 3. The time required for reorientation is $t_f=2.442$ s, and the residual energy is found to be $E=0.175$ Nm. The residual energy is diminished to some degree (20%) since the control for this example "rings" the structure less than the bang-bang control; however, the cost is a 14% increase in the time required to complete the maneuver.

Most of the residual energy is due to the excitation of the second mode, as shown in Fig. 4, and although this maneuver accomplishes the goal of controlling the rigid body mode, the flexural vibration may be too great for some applications.

Case 2: ($\alpha=0.95$, $\Delta\tau=0.1$)

For the second case, we shall reduce the sharpness of the switch function while maintaining the same rise time. By far, the largest impulse applied to the structure is the sudden change in sign of the control moment. Therefore, by reducing the control rate during the switch, a maneuver with less residual energy should result. The control profile for this case is shown in Fig. 5, with a final maneuver time of $t_f=2.695$ s and a total flexural energy of $E=0.276$ Nm at the completion of the maneuver.

The final flexural energy in this case is larger than the energy of the bang-bang maneuver, an increase of 16%, with an increase of 26% in the time required for the reorientation. Although the final residual energy is larger for this case, the peak energy imparted to the structure is less (0.578 Nm as compared to 1.381 Nm for the bang-bang maneuver). Considering the potential for exceeding maximum bending moments, the maneuver given by case 2 may be considered a better result for some applications even though the remaining structural vibrations will be larger. It is also evident that the frequency content of particular torque profiles manifests itself as spillover into the structure's residual modes in a complicated, sometimes counterintuitive way.

The range of the control parameters is quite extensive and, in fact, we can easily produce controls for the same boundary conditions with extraordinarily different character. For example, with $\alpha=0.99$ and $\Delta\tau=0.4$, the control is as shown in Fig. 6, where the final time is $t_f=3.599$ s and the flexural energy is 0.213 Nm. Since the interior switch is still sharp and the magnitude changes by $2u_{\max}$, the residual energy is quite high. However, as a consequence of the rise time requiring a full 40% of the total maneuver time, the minimum time is also quite large.

The last extreme variation that we shall make is to set $\alpha=0.95$ and $\Delta\tau=0.4$, resulting in the control shape shown in Fig. 7. This maneuver is quite smooth with little flexural excitation (the final energy is only 0.0322 Nm), but the final time is 4.071 s (a 90% increase over the bang-bang maneuver). This control profile resembles the results obtained from open-loop optimal control formulations with penalties applied to the control rates and control accelerations.[5]

Control of the Flexural Modes

Although the results presented in the preceding examples are rich in content with respect to optimizing the parameter

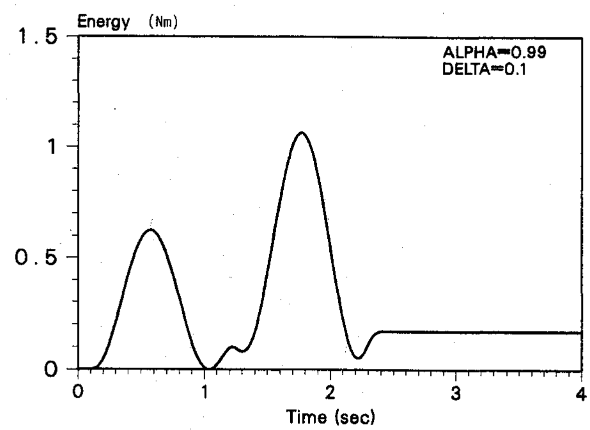

Fig. 4 Energy profile of mode 2.

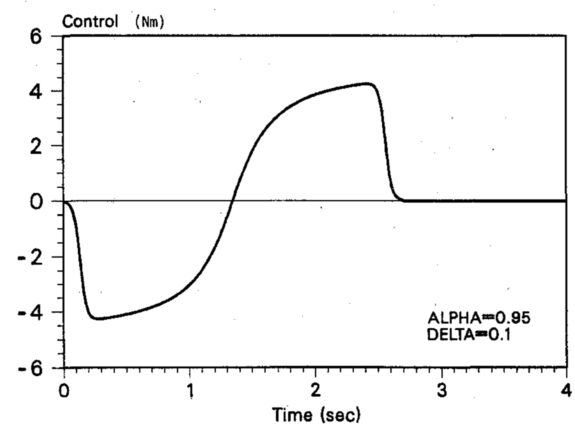

Fig. 5 Control for case 2.

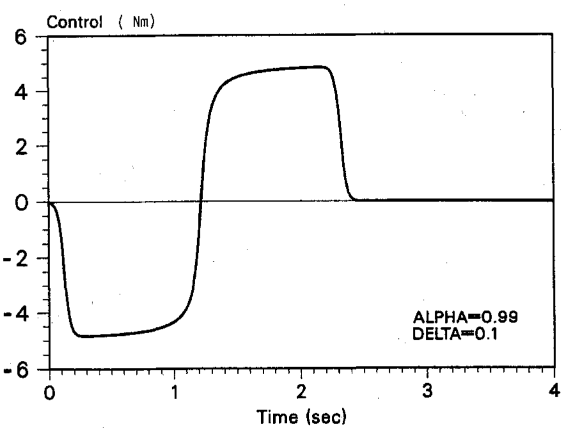

Fig. 3 Control profile for case 1.

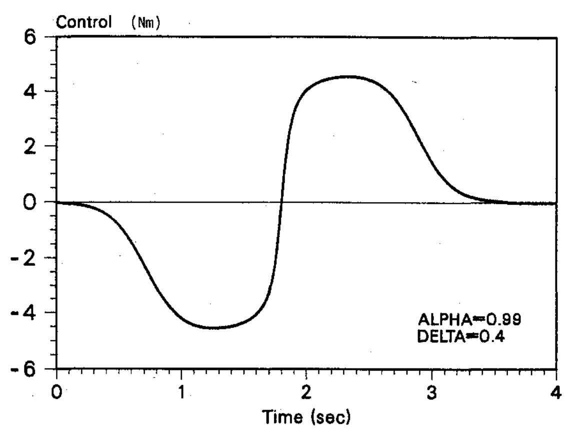

Fig. 6 Control for $\alpha=0.99$ and $\Delta\tau=0.4$.

selection for specific mission requirements (and our treatment of the subject has necessarily been brief), we shall now consider cases that explicitly include the first flexural mode in the minimum time formulation. Furthermore, we shall consider only a variation in the value of α, keeping $\Delta\tau$ fixed at 0.1 for simplicity of the discussion.

Case 3: $\alpha = 0.99$, $\Delta\tau = 0.1$

Setting $\alpha = 0.99$, the 5 deg rotation is accomplished in 2.473 s with a total residual energy of 0.0258 Nm. Note that this energy level is much lower than the final energy of case 1. The energy history of modes 1 and 2 combined are shown in Fig. 8, clearly indicating that both modes have been completely controlled.

The optimal control (shown in Fig. 9) switches three times during the course of the maneuver in order to control the first flexural mode. However, the control does not saturate after each switch. The two additional switches for this relatively stiff system are primarily for proper "phasing" of the controlled modes. Significantly different results have been obtained with models that are not as stiff and, consequently, have much lower natural frequencies.

For example, we have applied this method to a similar structure with a longer appendage and a reduced modulus of elasticity[8]; the frequency of mode 2 is 1.13 rad/s, and the frequency of mode 5 is only 11.86 rad/s. The near-minimum time approach (with $\alpha = 0.99$ and $\Delta\tau = 0.1$) produces a control profile as shown in Fig. 10. For this model, the control must not only correctly phase the modes but, due to the flexibility of the structure, must provide sufficient authority to eliminate the larger modal deformations that occur, resulting in near saturation of the control after each of the three switches.

Case 4: $\alpha = 0.95$, $\Delta\tau = 0.1$

By reducing the sharpness of the switches, we have demonstrated in case 2 that the resulting peak energy levels were greatly reduced, although the final energy was larger. We know that most of this residual energy is contained in the first flexural mode by examining the residual energy for each mode independently. By controlling the first two modes, the final residual energy can be reduced dramatically. Upon solving the near-minimum time problem for $\alpha = 0.95$ with control of the first flexural mode included, the maneuver time is increased to 2.744 s; however, the residual energy at the final time is reduced to only 0.0122 Nm. Clearly, this is the best of the four cases we have examined in terms of residual energy with a modest increase in the maneuver time. The control profile resembles the result for case 3 and is therefore not shown graphically. The control for case 4 also exhibits three switches as found in case 3, but the magnitude of the change in the control is quite small by comparison, a direct cause of the low residual energy.

Case 5: $\alpha = 0.95$, $\Delta\tau = 0.1$

In the final case to be examined, we control modes 1–3 of the structure for the control parameters $\alpha = 0.95$ and $\Delta\tau = 0.1$. Cases 3 and 4 demonstrated that residual energy levels can be reduced significantly without paying a high price in maneuver time; controlling three modes produces further evidence of this truth. The results obtained are quite dramatic. The maneuver time is 28% longer (2.752 s) than the bang-bang case; however, the residual energy is essentially zero (0.227×10^{-5} Nm). The control profile (Fig. 11) has the same basic character as found in case 4, but there is an additional

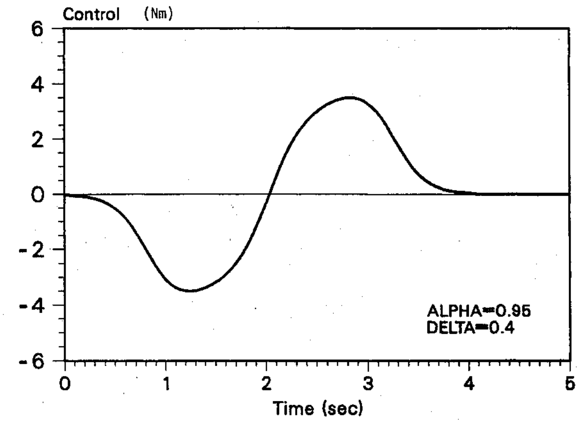

Fig. 7 Control for $\alpha = 0.95$ and $\Delta\tau = 0.4$.

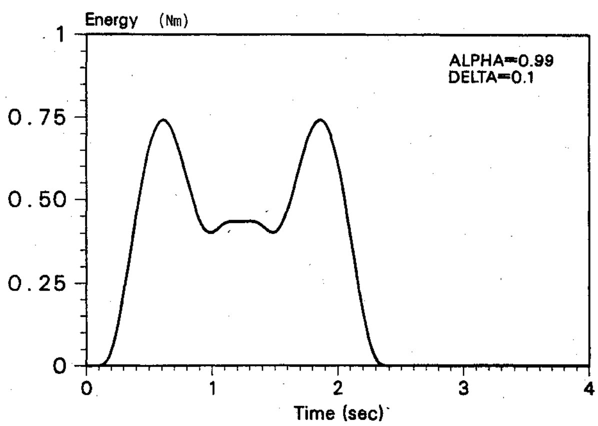

Fig. 8 Energy history of modes 1 and 2.

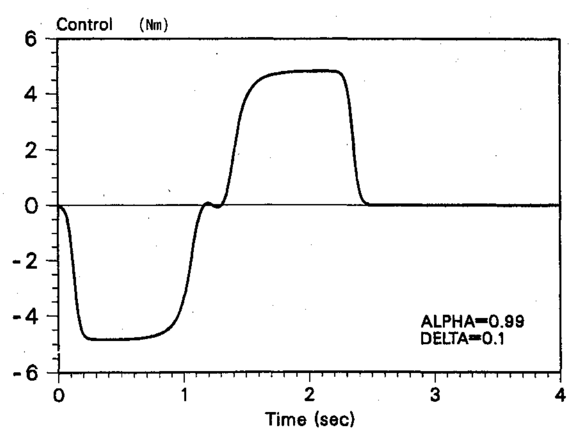

Fig. 9 Control for case 3.

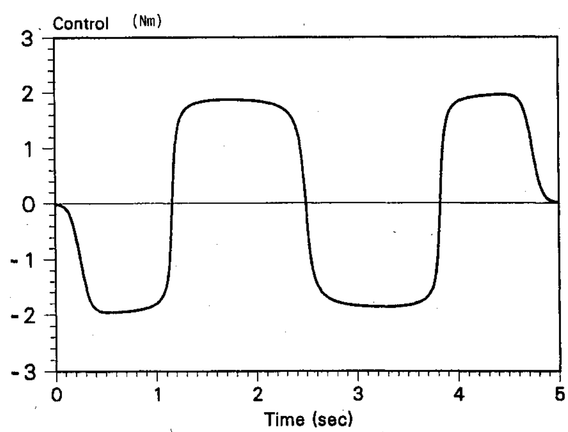

Fig. 10 Control profile for a highly flexible body.

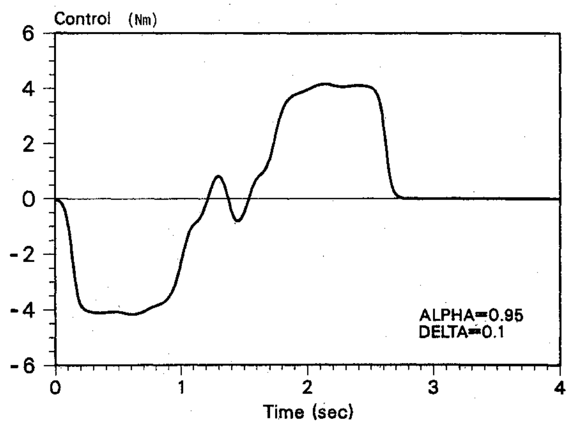

Fig. 11 Control for case 5.

higher frequency content included. Again, the control switches three times but does not saturate after each switch.

Conclusions

We have presented a versatile, near-minimum time, open-loop optimal control formulation for flexible structures. The applied control is implicitly saturation-bounded and is parameterized by two independent and arbitrary constants such that the control can be shaped to meet specific hardware constraints or mission requirements. Solution of the two-point boundary value problem is usually rapid, requiring four to six iterations for a low number of controlled modes.

As a consequence of the ability to shape the controls, we have a convenient means of evaluating specific tradeoffs with respect to manevuer time and residual energy as a function of the degree of control smoothness. For a series of solutions using the minimum time formulation with various values of the control parameters, surface plots can be produced giving excellent estimates of both the maneuver time and residual energy as a function of the independent "torque-shaping" variables. Taking this concept one step further, it is easy to see that an a priori computed family of maneuvers can be parameterized to establish near-minimum time optimal controls as a function of boundary conditions and maneuver time, eliminating the need to solve nonlinear ODE's for each case. The near-minimum time formulation represents a practical and realistic approach to solving open-loop, minimum time slewing maneuvers.

References

[1]Vander Velde, W.E. and He, J., "Design of Space Structure Control Systems Using On-Off Thrusters," *Journal of Guidance, Control, and Dynamics,* Vol. 6, Jan.–Feb. 1983, pp. 53–60.

[2]Roberts, S.M. and Shipman, J.S., *Two-Point Boundary Value Problems: Shooting Methods,* American Elsevier, New York, 1972.

[3]Miele, A., "Method of Particular Solutions for Linear, Two-Point Boundary Value Problems," *Journal of Optimization Theory and Applications,* Vol. 2, July, 1968, pp. 260–273.

[4]Vadali, S.R., "Solution of the Two-Point Boundary Value Problems of Optimal Spacecraft Rotational Maneuvers," Ph.D. Dissertation, Virginia Polytechnic Institute and State Univ., Blacksburg, VA, 1982.

[5]Junkins, J.L. and Turner, J.D., *Optimal Spacecraft Rotational Maneuvers,* Elsevier Science Publishers B.V., Amsterdam, Netherlands, 1986.

[6]Subrahmanyam, M.B., "Computation of Optimal Controls by Newton's Method Using a Discretized Jacobian," *Journal of Guidance, Control, and Dynamics,* Vol. 9, May–June 1986, pp. 371–374.

[7]Meirovitch, L., *Computational Methods in Structural Dynamics,* Sijthoff & Noordhoff, Rockville, MD, 1980.

[8]Thompson, R.C., Junkins, J.L., and Vadali, S.R., "Near Minimum Time Open Loop Control of Dynamic Systems," AIAA Paper 87-0958, April 1987.

[9]Kirk, D.E., *Optimal Control Theory; An Introduction,* Prentice-Hall, Englewood Cliffs, NJ, 1970.

[10]Bryson, A.E., Jr. and Ho., Y.-C., *Applied Optimal Control,* Hemisphere, Washington, DC, 1975.